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LMI-Based Model Reduction for a Vectored-Thrust Ducted Fan

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Abstract

This paper contains the first experimental application of model reduction methods developed for systems represented by a Linear Fractional Transformation (LFT) on a repeated scalar uncertainty structure. These methods provide for reduction of both the state order, and the uncertainty descriptions. LFT models of a vectored-thrust flight control experiment are reduced in order to make controller synthesis feasible. Recent Linear Parameter Varying (LPV) design techniques are used to synthesize the controllers.

1 Introduction

Systematic methods which provide for reduction of both the state order and uncertainty descriptions of uncertain system realizations have recently been developed [2], [3]. These methods rely on the solution of two Linear Matrix Inequalities (LMIs), which generalize the system Lyapunov equations associated with standard state-space realizations. Furthermore, *a priori* guaranteed error bounds for the reduction are easily computed. In this paper we present the first experimental application of these LMI-based model reduction methods.

We consider the reduction of two separate Linear Fractional Transformation (LFT) models for a vectored-thrust flight control experiment [6]. Due to the complexity of the full-sized models, numerical difficulties arise in the controller synthesis process, and as a result Linear Parameter Varying (LPV) controller designs can not be completed. Using the LMI-based model reduction methods, we are able to obtain reduced models for which we can design LPV controllers for the experiment. The LPV controllers for the reduced models will be compared to the controllers discussed in [5] and [10].

The vectored-thrust flight control experiment is discussed in Section 2, including a review of the derivation of the LFT models and the controller design process. The model reduction procedure for uncertain systems is presented in Section 3. Model reduction results are presented in Section 4.

2 The Ducted Fan Experiment

The intent of the flight control experiment is to have a simple aircraft with two-dimensional vectored and reverse thrust. The aircraft, a ducted fan engine, is bolted to a rotating arm, which limits motion to three degrees of freedom: one rotational and two translational, approximately on the surface of a sphere defined by the arm. With this geometry, the ducted fan is completely controllable with just the vectored thrust. Flaps on the fan allow the thrust to be vectored from side to side and even reversed. In [6], a detailed description of the fan is given, including models for the thrust as a function of flap angle and fan speed, as well as a discussion of ground effects. The software interface for the controllers is discussed in [5].

2.1 Modelling the Ducted Fan

A nonlinear state space model for the ducted fan is constructed using first principles analysis based on standard rigid body mechanics. A six state model, $(\alpha_1, \alpha_2, \alpha_3, \dot\alpha_1, \dot\alpha_2, \dot\alpha_3)$, was selected for the control designs, where α_1 corresponds to horizontal translation of the fan, α_2 to vertical translation, and α_3 to pitch angle. The equations of motion are derived from Lagrange's equations for the system [4]. The resulting nonlinear state space realization matrices have the form

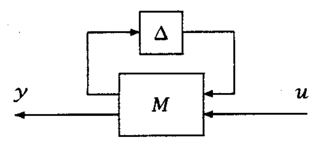

Figure 1: LFT System

$\{A(\alpha), B(\alpha), C(\alpha), D(\alpha)\}$, where the coefficients of these matrices vary as nonlinear functions of α. The model is fairly accurate, although simplifications have been made (see [5]).

Examination of the trajectories run on the experiment reveals that the most relevant variations in the realization matrices coefficients occur as functions of α_3 and $\dot\alpha_1$. Although the rates, such as $\dot\alpha_1$, are not measured, an inner software control loop estimates them; hence in our models $C(\alpha) = I$. Moreover, the fan is strictly proper, thus $D(\alpha) = 0$. As a result, $A(\alpha)$ and $B(\alpha)$ are the only matrices which have parameter variations. The structure of these matrices is $[\,A(\alpha)|B(\alpha)\,] =$

$$\left[\begin{array}{cccccc|cc} 1 & 0 & 0 & T & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & T & 0 & 0 \\ 0 & a_{42}(\alpha_3,\dot\alpha_1) & a_{43}(\alpha_3) & 1 & 0 & 0 & b_{41}(\alpha_3) & b_{42}(\alpha_3) \\ 0 & a_{52}(\dot\alpha_1) & a_{53}(\alpha_3) & 0 & 1 & 0 & b_{51}(\alpha_3) & b_{52}(\alpha_3) \\ 0 & a_{62}(\alpha_3,\dot\alpha_1) & a_{63}(\alpha_3) & 0 & 0 & 1 & b_{61}(\alpha_3) & b_{62}(\alpha_3) \end{array}\right] \quad (1)$$

where T is the sampling rate. The state feedback nature of the problem is not currently exploited in the designs.

Using the nonlinear state space models, we construct a LFT representation for the ducted fan. The LFT paradigm is represented pictorially in Figure 1, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a constant realization matrix, and Δ contains: copies of the delay operator q; representations of nonlinear behavior that can not be approximated by LTI models; and structured uncertainty. The mapping $u \mapsto y$ for this system is given by the LFT

$$y = (\Delta \star M)u = (D + C\Delta(I - A\Delta)^{-1}B)u,$$

where $u, y \in l_2$. We refer to such models by the pair $(\boldsymbol{\Delta}, M)$; for more details on the LFT framework see, e.g., [7] or [8].

The Δ structure we use for the ducted fan is

$$\boldsymbol{\Delta} = \left\{\mathrm{diag}\left[\delta_1 I_{n_1}, \delta_2 I_{n_2}, \delta_3 I_{n_3}\right] : \delta_i \in \mathcal{L}(l_2)\right\}, \quad (2)$$

where $\delta_1 := q$, $\delta_2 := \dot\alpha_1$, and $\delta_3 := \alpha_3$. To place the parameter dependence into the LFT framework, each of the coefficients in (1) is fit with a rational function using a least-squares technique. The dimensions n_1, n_2 and n_3 of the Δ structure subblocks depend on the order of the fit used.

We often consider Δ which lie in a norm-bounded subset of $\boldsymbol{\Delta}$, i.e., $\mathbf{B}\boldsymbol{\Delta} = \left\{\Delta \in \boldsymbol{\Delta} : \|\Delta\|_{l_2 - l_2} \le \beta\right\}$, where $\beta > 0$ and $\|\cdot\|$ denotes the l_2-induced norm. For a given uncertainty set, $\boldsymbol{\Delta}$, we denote the commuting matrix set by $\mathcal{T} = \{T \in \mathbb{C}^{N\times N} : T\Delta = \Delta T,\ \forall \Delta \in \boldsymbol{\Delta}\}$. For $\boldsymbol{\Delta}$ as defined in (2), $T \in \mathcal{T}$ if $T = \mathrm{diag}[T_1, T_2, T_3]$, where each $T_i \in \mathbb{C}^{n_i \times n_i}$.

2.2 LPV Controller Synthesis

Recent theoretical and computational machinery for gain-scheduling has led to new design techniques which provide *a priori* guarantees on the performance and/or stability of the resulting closed-loop system. These techniques produce an optimal parameter-dependent controller (an LPV controller), where the parameters are the same measurable δ_i's that are used in the LFT model for the plant, e.g., for the ducted fan, both Δ_P and Δ_K in Figure 2 contain copies of δ_1, δ_2 and δ_3. These controllers are essentially gain-scheduled controllers based on a continuous set of linearizations of the model; The controller is

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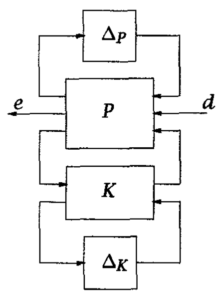

Figure 2: Parameter Varying System

optimized to provide performance against the worst-case time variation of the parameters. See [12], [1] and [5] for details.

The model P and the Δ_P structure are augmented to collect all the parameters and states together; the resulting Δ structure is diag$\{\Delta_P, \Delta_K\}$, and the controller K is then treated as a simple matrix. The resulting control problem is a *robust* control problem with additional structure on the plant and parameters. The design objective is to find K such that the interconnection is stable and the l_2-induced norm from d to e is small for all allowable parameter variations Δ. Note that to synthesize a controller for a system, the given model must satisfy the stabilizability and detectability conditions of [11].

3 LMI-based Model Reduction

For standard realizations the role of Lyapunov equations, and controllability and observability gramians, in model reduction are well known (see e.g., [9]). In [2] it is shown that a general version of these concepts hold for LFT realizations with uncertainty descriptions incorporated into the model. A summary of these results is presented here.

Given a realization $(\mathbf{\Delta}, M)$ and any $\epsilon \geq 0$, a lower order realization $(\mathbf{\Delta}_r, M_r)$ exists such that the l_2-induced norm of the difference between the full and reduced realizations is bounded by ϵ if and only if there exist structured rank-constrained solutions to a pair of LMIs. For example, for the ducted fan models with $\mathbf{\Delta}$ defined as in (2), a lower order realization would have $\mathbf{\Delta}_r = \left\{\text{diag}\left[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \delta_3 I_{r_3}\right] : \delta_i \in \mathcal{L}(l_2)\right\}$, with $r_i \leq n_i$ for $i = 1, 2, 3$. The main model reduction result is stated below [2].

Theorem 1 *Given a realization $(\mathbf{\Delta}, M)$, with $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then there exists a reduced realization $(\mathbf{\Delta}_r, M_r)$ such that $\sup_{\Delta \in \mathbf{B\Delta}} \|(\Delta \star M) - (\Delta_r \star M_r\| \leq \epsilon$ if and only if there exists $X > 0$ and $Y > 0$, both in $\mathcal{T}$, satisfying*

(i) $AXA^* - X + BB^* < 0$

(ii) $A^*YA - Y + C^*C < 0$, and

(iii) $\lambda_{min}(XY) = \epsilon^2$, with multiplicity $\sum_i (n_i - r_i)$

where $\epsilon > 0$.

To derive reduction error bounds we apply Theorem 1 recursively to a *balanced* realization, i.e., a realization where solutions to the Lyapunov inequalities in (i) and (ii) above are found such that $Y = X = \Sigma$, with $\Sigma > 0$ and diagonal. We partition the system matrices A, B, C and Σ conformally with the block structure Δ, then we partition each block of Σ based on where we will truncate the realization; i.e., let $\Sigma_i = \text{diag}[\hat{\Sigma}_{1i}, \Sigma_{2i}]$ for $i = 1, \ldots, p$, where $\hat{\Sigma}_{1i} = \text{diag}[\sigma_{i1} I_{s_{i1}}, \cdots, \sigma_{ik_i} I_{s_{ik_i}}]$, and $\Sigma_{2i} = \text{diag}[\sigma_{i(k_i+1)} I_{s_{i(k_i+1)}}, \cdots, \sigma_{it_i} I_{s_{it_i}}]$, $k_i \leq t_i$. The submatrices of A, B and C corresponding to Σ_{2i} are truncated. The following balanced truncation model reduction result for uncertain systems follows directly from Theorem 1.

Corollary 1 *Suppose $(\mathbf{\Delta}_r, M_r)$ is the reduced model obtained from the balanced system $(\mathbf{\Delta}, M)$. Then*

$$\sup_{\Delta \in \mathbf{B\Delta}} \|(\mathbf{\Delta} \star M) - (\mathbf{\Delta}_r \star M_r)\| \leq \sum_{i=1}^{p} \sum_{j=k_i+1}^{t_i} \sigma_{ij}. \qquad (3)$$

4 Model Reduction Results

We reduced two LFT models for the ducted fan using the LMI based model reduction techniques; these are easily implemented using existing convex optimization algorithms.

In the first model, which we denote by $P1_F$, only some of the parameter variations of Equation (1) are included. The dependence of a_{42} and a_{62} upon $\dot{\alpha}_1$ was ignored, and these parameters were modelled as lines depending on α_3 only; a_{52} was modelled as a quadratic depending on $\dot{\alpha}_1$, while the other parameters in the A matrix were modelled as second order LFTs. In the B matrix, the parameters b_{41}, b_{52}, and b_{61} were approximated as lines, while b_{42} and b_{62} were treated as quadratics. The parameter b_{51} was held constant at the value taken by the linearization of the model around hover. The resulting LFT model for $P1_F$ has a Δ structure with $n_1 = 6$, $n_2 = 2$, and $n_3 = 13$. For the second model, which we denote by $P2_F$, there is one more addition; we now model a_{42} as a surface that depends on both α_3 and $\dot{\alpha}_1$. The resulting LFT model has a Δ structure with $n_1 = 6$, $n_2 = 4$, and $n_3 = 16$. In both models the range of values for the parameters $\dot{\alpha}_1$ and α_3 correspond to forward flight.

For each of the models $P1_F$ and $P2_F$, we obtained two reduced models for which we were able to synthesize LPV controllers. The reduction results are shown in Table 4.

Model	$\Delta : [n_1,\ n_2,\ n_3]$	Error Bound
Full Model: $P1_F$	[6, 2, 13]	
Reduced Model: $P1_{Ra}$	[6, 2, 5]	0.2234(3%)
Reduced Model: $P1_{Rb}$	[6, 2, 3]	0.7026(8%)
Full Model: $P2_F$	[6, 4, 16]	
Reduced Model: $P2_{Ra}$	[6, 2, 4]	0.5144(6%)
Reduced Model: $P2_{Rb}$	[6, 2, 3]	0.7467(9%)

Table 1: Ducted Fan Reduction Results

LPV controllers have been synthesized using all four reduced models. Although the reduction error is greater for $P1_{Rb}$ and $P2_{Rb}$, the predicted gain of the closed loop system from d to e is smaller using the LPV controllers designed with these models. Experimental assessment of the controllers is currently being completed.

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